System of linear differential Equations

• homogenous system

$$\dot{x}(t) = Ax(t) \quad , \; x(0) = x_0$$

• Solution

$$x(t) = e^{At} x_0$$

$$e^{At} = T e^{Dt} T^{-1}$$

• non-homogenous

$$\dot{x}(t) = Ax(t) + u(t)$$

Control vector

• Solution

$$x(t) = e^{At} x_0 + e^{At} \int_0^t e^{-A\tau} u(\tau) d\tau$$

Stability of system

• naturally stable

all $Re\, \lambda_i \leq 0$

• unstable

at least one eigenvalue $Re\, \lambda_i > 0$

• خطوات الحل

1- نحسب الـ eigenvalues

2- نحسب الـ eigenvectors

3- نحسب e^{At}

4- نعوض في الـ Form of Solution

Elemenation by differentiation

تعتمد هذه على تحويل الـ system لـ "2^{nd} order d. eq^{n}" في متغير واحد فقط وحلها كما سبق دراسته.

Remember

$$x(t) = x_h(t) + x_p(t)$$

شكل الـ eigenvalues	$x_h(t)$
1- real and not repeated $(\lambda_1, \lambda_2, ---)$	$x_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + ----$
2- real and repeated $\lambda = \lambda_1 = \lambda_2 = ---$	$x_h(t) = (c_1 + c_2 t + c_3 t^2 + ----) e^{\lambda t}$
3- complex and not rep. $\lambda_i = \alpha_i \pm i\beta_i$	$x_h(t) = (c_1 \cos\beta t + c_2 \sin\beta t) e^{\alpha t}$
4- Complex and repeated	$x_h(t) = \begin{bmatrix} (c_1 \cos\beta t + c_2 \sin\beta t) + \\ t(c_3 \cos\beta t + c_4 \sin\beta t) + --- \end{bmatrix} e^{\alpha t}$

ملحوظة

- لحل المعادلة وحساب الـ $x_h(t)$ نضع الـ $u(t)=0$ ثم بعد ذلك نحسب الـ $x_p(t)$ الحل الخاص بالجزء الـ non-homogeneous

- يتم حساب الـ $x_p(t)$ عن طريق الـ D-operator method

$$x_p(t) = \frac{1}{L(D)} \cdot u(t)$$

$u(t)$	$x_p(t) = \frac{1}{L(D)} \cdot u(t)$
1- $e^{\alpha t}$	• $x_p(t) = \frac{1}{L(\alpha)} \cdot e^{\alpha t}$, $L(\alpha) \neq 0$ • $x_p(t) = e^{\alpha t} \cdot \frac{1}{L(D+\alpha)} \cdot 1$, $L(\alpha) = 0$ ملحوظة في حالة وجود صفر في المقام عند $D=\alpha$ نحلل المقام ونعوض في الأقواس التي لا يوجد بها مشكلة ثم نستخدم قاعدة الإزاحة.
2- $e^{\alpha t} \cdot F(t)$	$x_p(t) = e^{\alpha t} \cdot \frac{1}{L(D+\alpha)} \cdot F(t)$
3- $\sin \alpha t$ or $\cos \alpha t$	$x_p(t) = \frac{1}{L(-\alpha^2)} \cdot \sin \alpha t$ or $\cos \alpha t$, $L(-\alpha^2) \neq 0$ put $\frac{1}{D^2+\alpha^2} \cdot \sin(\alpha t) = \frac{-t}{2\alpha} \cos(\alpha t)$, $L(-\alpha^2) = 0$ $\frac{1}{D^2+\alpha^2} \cdot \cos(\alpha t) = \frac{t}{2\alpha} \sin(\alpha t)$
4- t^n or polynomial	نستخدم نظرية ذات الحدين $(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \cdots$
5- $\sinh \alpha t$ or $\cosh \alpha t$	$\sinh \alpha t = \frac{e^{\alpha t} - e^{-\alpha t}}{2}$ $\cosh \alpha t = \frac{e^{\alpha t} + e^{-\alpha t}}{2}$ ثم نستخدم قانون الإزاحة لـ "e".

1- Find the matrix A whose eigenvalues 1, 4 and whose eigenvectors are $\begin{bmatrix}3\\1\end{bmatrix}$, $\begin{bmatrix}2\\1\end{bmatrix}$ respectively.

Sol.

$$A = T D_\lambda T^{-1}$$

$$T = \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$$

$$D_\lambda = \begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}$$

$$A = \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix} = \begin{bmatrix}3 & 8\\1 & 4\end{bmatrix}\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$$

$$A = \begin{bmatrix}-5 & 18\\-3 & 10\end{bmatrix}$$

2- Decide the stability of the system

1- $\dfrac{dx}{dt} = \begin{bmatrix}0 & 4\\-1 & 0\end{bmatrix} x$

Sol.

$$\begin{vmatrix}-\lambda & 4\\-1 & -\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4$$

$$\lambda = \pm 2i$$

System naturally stable.

2- $\dfrac{dx}{dt} = \begin{bmatrix}3 & 1\\1 & 3\end{bmatrix} x$

Sol.

$$\begin{vmatrix}3-\lambda & 1\\1 & 3-\lambda\end{vmatrix} = 0 \Rightarrow (3-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda-2)(\lambda-4) = 0 \Rightarrow \lambda = 2, 4$$

System unstable

3- Solve the Following system

a- $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix}$

Sol.

• For homogenous solution.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

• eigen values

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 3 \\ -1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(-1-\lambda)+3=0$$

$$\therefore \lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda-2) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 2$$

• eigen vectors

$\lambda = 0$

$$(A-\lambda I)x_1 = \square \Rightarrow \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{3R_2+R_1}$$

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & -1 & 0 \end{array}\right) \equiv \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \equiv \begin{array}{c} \begin{array}{cc} I & q \end{array} \\ \left(\begin{array}{c|c|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array}$$

$$\therefore x_1 = c\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda = 2$

$$(A-\lambda I)x_2 = \square \Rightarrow \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} x_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{R_1+R_2}$$

$$\begin{array}{c} \begin{array}{cc} I & q \end{array} \\ \left(\begin{array}{c|c|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array} \Rightarrow x_2 = c\begin{pmatrix} 3 \\ -1 \end{pmatrix} \Rightarrow x_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

• e^{At}

$$T = \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix} \Rightarrow T^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$$

$$e^{D_\lambda t} = \begin{pmatrix} e^{0t} & 0 \\ 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2t} \end{pmatrix}$$

$$\therefore e^{At} = T e^{D_1 t} T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 3e^{2t} \\ -1 & -e^{2t} \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1+3e^{2t} & -3+3e^{2t} \\ -1-e^{2t} & 3-e^{2t} \end{pmatrix}$$

$$\therefore x_h(t) = \frac{1}{2}\begin{pmatrix} -1+3e^{2t} & -3+3e^{2t} \\ -1-e^{2t} & 3-e^{2t} \end{pmatrix} x_0$$

Non-homogenous Solution

$$u(t) = e^{At}\int_0^t e^{-A\tau} \cdot u(\tau)\, d\tau$$

$$= \frac{1}{4}\int_0^t \begin{pmatrix} -1+3e^{-2\tau} & -3+3e^{-2\tau} \\ -1-e^{-2\tau} & 3-e^{-2\tau} \end{pmatrix} \cdot \begin{pmatrix} \tau \\ 0 \end{pmatrix} d\tau$$

$$= \frac{1}{4}\int_0^t \begin{pmatrix} \tau(-1+3e^{-2\tau}) \\ \tau(-1-e^{-2\tau}) \end{pmatrix} d\tau$$

$u=\tau \quad dv = -1+3e^{-2\tau}$

$du = d\tau \quad v_1 = -\tau - \frac{3}{2}e^{-2\tau}$

$d^2u = 0 \quad v_2 = -\frac{\tau^2}{2} + \frac{3}{4}e^{-2\tau}$

$$= \frac{1}{4}\left[\begin{pmatrix} \tau(-\tau-\frac{3}{2}e^{-2\tau})\Big|_0^t & (-\frac{\tau^2}{2}+\frac{3}{4}e^{-2\tau})\Big|_0^t \\ \tau(-\tau+\frac{e^{-2\tau}}{2})\Big|_0^t & (-\frac{\tau^2}{2}-\frac{e^{-2\tau}}{4})\Big|_0^t \end{pmatrix}\right]$$

$u=\tau \quad dv = -1-e^{-2\tau}$

$u' = d\tau \quad v_1 = -\tau + \frac{e^{-2\tau}}{2}$

$u'' = 0 \quad v_2 = -\frac{\tau^2}{2} - \frac{e^{-2\tau}}{4}$

$$u(t) = \frac{1}{4}\begin{pmatrix} t(-t-\frac{3}{2}e^{-2t}) & (-\frac{t^2}{2}+\frac{3}{4}e^{-2t}-\frac{3}{4}) \\ t(-t+\frac{e^{-2t}}{2}) & (-\frac{t^2}{2}-\frac{e^{-2t}}{4}+\frac{1}{4}) \end{pmatrix}$$

$$\therefore x(t) = x_h(t) + u(t)$$

b- $\dot{x} = y + t$ — ① , $\dot{y} = -2x + 3y + 1$ — ②

Sol.

• For eqn. ① D.r.t

$$\ddot{x} = \dot{y} + 1$$

• From ②

$$\ddot{x} = -2x + 3y + 2$$

• From ①

$$y = \dot{x} - t \quad — ③$$

$$\therefore \ddot{x} = -2x + 3\dot{x} - 3t + 2 \Rightarrow \boxed{\ddot{x} - 3\dot{x} + 2x = -3t + 2}$$

$$\therefore x(t) = x_h(t) + x_p(t)$$

• $x_h(t)$

$$\therefore \lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 2)(\lambda - 1) = 0 \Rightarrow \lambda_1 = 1 , \lambda_2 = 2$$

$$\therefore x_h(t) = c_1 e^{t} + c_2 e^{2t}$$

• $x_p(t)$

$$(D^2 - 3D + 2)\, x_p(t) = -3t + 2 \Rightarrow x_p(t) = \frac{1}{(D^2 - 3D + 2)} \cdot (-3t + 2)$$

$$\therefore x_p(t) = \frac{1}{2} \cdot \left(1 + \frac{D^2 - 3D}{2}\right)^{-1} \cdot (-3t + 2)$$

$$= \frac{1}{2}\left(1 - \cancel{\frac{D^2}{2}} + \frac{3D}{2} + \cancel{\left(\frac{D^2 - 3D}{2}\right)^2} \text{-----}\right)(-3t + 2)$$

$$0 \qquad 0$$

$D(-3t+2) = -$

$D^2(-3t+2) =$

$$x_p(t) = \frac{1}{2}\left(-3t + 2 - \frac{9}{2}\right) = \frac{1}{2}\left(-3t - \frac{5}{2}\right)$$

$$\therefore x(t) = c_1 e^{t} + c_2 e^{2t} - \frac{1}{2}\left(3t + \frac{5}{2}\right)$$

• From ③

$$y(t) = c_1 e^{t} + 2c_2 e^{2t} - \frac{3}{2}$$